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PORTFOLIO SELECTION AND THE CAPITAL ASSET PRICING MODEL

What portfolio would you recommend to a 28-year-old who has just been promoted to a management position, and what portfolio would you recommend to a 60-year-old who has just retired?

The perennial question in the world of personal finance seems to be “What should I buy?” A far better question would be “What portfolio should I hold?” The research exercise described in this note provides an opportunity to explore the reasoning behind this distinction. More importantly, it will become apparent that the answers to the portfolio question provide critical insights into how we measure risk and determine appropriate rates of return for a given level of risk. In particular, the analysis described here will develop familiarity with the capital asset pricing model (CAPM), a model of appropriate returns based on the relation between the returns on an individual asset and the returns on a broad market portfolio.

The analysis described here is organized around the question stated at the top of this page. To focus our analysis, we will ignore issues related to the amount of wealth these individuals might have, the tax situation each may face, and specific expenditure plans they might have. Simply assume that both have relatively large amounts to invest, but not so large that they are indifferent to the returns they might earn. Both would, of course, prefer higher returns to lower returns. The newly promoted manager, however, would be able to accept a higher degree of risk than the retiree.

The analysis will proceed as follows. The accompanying spreadsheet (UVA-F-1604X) presents historic returns of three stocks as well as the returns on the S&P 500 index. You will be guided through the analysis of these data. Each analysis will be accompanied by a series of questions. The first set of steps examines various portfolios of investments; the second set explores the implications of the portfolio results for individual stocks.

Portfolio Selection

We will begin this analysis by considering just single investments and then proceed to add additional alternatives, including portfolios combining individual investments.[1] The analysis will make use of basic statistical descriptions of returns and employ a very simple regression analysis.

Step 1: Calculate the realized returns for a hypothetical portfolio invested equally in the three stocks. You may simply average the three returns.[2] We will refer to this as the equally weighted portfolio. Calculate the mean (average) realized return and standard deviation of the realized returns for each stock, the S&P 500 index, and the equally weighted portfolio.

- Compare and contrast the mean and standard deviations of these five possible investments. What are the noticeable differences?
- When choosing between the three individual stock investments and ignoring combinations of these investments, which would you suggest to our hypothetical investors? Why?
- How does your answer change if you add the index as an allowable investment?

Step 2: Calculate the correlation between each of the three stocks and between the S&P 500 and the equally weighted portfolio. Assume that the average return, standard deviation and correlation of each of these assets from 1990 to 2009 are a good estimate of these same measures for the future. Using the template provided in the second tab, construct a graph that shows the expected future relation between the returns (vertical axis) and standard deviations (horizontal axis) of a pair of the stock investments (your choice which pair).[3]

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[1] A portfolio can consist of a single investment and need not have more than one component.

[2] Averaging the returns implicitly assumes that each month, you are rebalancing the portfolio so that an equal value is invested in each stock.

[3] The template produces the graph by first generating a set of observations and then plotting those observations. This is done as follows. A table was created that has one column with the weight assigned to one stock (using increments of 1%); a second column with the weight assigned to the second stock (one minus the weight of the first stock); a third column with the anticipated standard deviation of this two-asset portfolio based on the weights, standard deviations, and correlation. The formula for the mean would be $r_p = w_1 \overline{r_1} + w_2 \overline{r_2}$ where r_p is the portfolio return, $\overline{r_1}$ is the mean historic return of stock number one, $\overline{r_2}$ is the mean historic return of stock number two, and *w* refers to the proportional weight invested in each subscripted security. For example, if 30% of a portfolio is invested in security number one, then $w_1 = 0.30$ and, since all weights must sum to 1, w_2 would equal 0.70. For the standard deviation, the formula would be $\sigma_p = sqrt\,(w_1^2 \sigma_1^2 + w_2^2\, std_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2\, corr_{1,2})$ where σ denotes the standard deviation of historic returns for the subscripted security, *corr* denotes the correlation between the subscripted securities, and the *p* subscript indicates the portfolio you are examining. The resulting expected returns are plotted against the standard deviations.

- Is there an intuitive explanation for the shape of the curve you have generated?
- How does the shape of the curve change as you alter the correlation (which is bounded by −1 and 1, inclusive)?
- What do the correlations between the individual stocks indicate? What insight is provided by the correlation between the equally weighted portfolio and the S&P 500 index?

Step 3: Using the template provided, construct a graph that shows the expected returns and standard deviations of all possible combinations of the three stocks. This graph was constructed in a fashion analogous to the two-asset graph, but expanded to three stocks.[4] This graph describes the complete investment space offered by the three stock investments.

- Restricting yourself to combinations of the three stocks, what portfolio of stock investments would you recommend for our two hypothetical investors?
- Locate the S&P 500 index on this graph. Explain the location of this index relative to the set of possible three-stock combinations.

Step 4: We can expand the set of financial assets by considering a risk-free bond (U.S. government security). We will assume that the return on the bond is 0.45% a month, which is about 5.5% per year.

- What would be the diversification benefits from adding a bond?

This risk-free asset can be combined with the market portfolio to yield another set of possible investments. This line is referred to as the capital market line. Add the capital market line to your graph from Step 3. The can be done by adding points to the graph that represent linear combinations of the bond and the market.

- Explain the shape of the capital market line.
- Now consider all the possible investments you have identified (the three individual stocks, all combinations of the three stocks, the index, and the security market line). What would you recommend to our two hypothetical investors? Be very specific in your recommendation.

[4] The formula for the mean would be $r_p = w_1 \bar{r}_1 + w_2 \bar{r}_2 + w_3 \bar{r}_3$ and for the standard deviation for three stocks would be $\sigma_p = sqrt\,(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_1\sigma_2 corr_{1,2} + 2w_1w_3\sigma_1\sigma_3 corr_{1,3} + 2w_2w_3\sigma_2\sigma_3 corr_{2,3})$. Given the number of possible combinations, the graph uses increments of 5% in the weights and starts by generating all 5% incremented combinations of weights on the three stock investments.

Individual Asset Returns

It should be clear from the previous steps that portfolios of stocks offer significant advantages over individual stocks. In fact, a case could be made that an extremely broad portfolio (a market index) is the optimal portfolio of risky investments, and this portfolio would be combined with an appropriate amount of riskless bonds to achieve a desired optimal overall investment portfolio.

The optimality of holding portfolios has important implications for individual stocks. When considering the riskiness of an *individual* stock, investors will ignore the total risk of the stock (since much of it will be diversified away) and consider only that level of risk that cannot be diversified away. This can be described as the risk an individual stock adds to a particular portfolio. The remaining steps develop a measure of this risk and link that measure to returns.

Step 5: To measure the risk a stock adds to a given portfolio, one simply regresses the returns of the individual stock on the returns of the portfolio. The regression coefficient on the portfolio reflects exactly that risk. For example, if we regress an individual stock's returns on a broad market index, the resulting coefficient on the market is called a market beta (or, more commonly, simply the beta). The beta reflects the variation in individual stock returns that cannot be diversified away. One simple form for the regression is the following:[5]

$$r_{i,t} = alpha + beta\ r_{m,t} + \varepsilon_t \tag{1}$$

In the regression model above, returns are observed for each time period *t* for an individual stock *i* and market return *m*. The regression acknowledges individual error terms each period of ε since stock returns will deviate from this relation due to firm-specific events. Please calculate the alpha and beta for each of the three stocks relative to the S&P 500 index. This can be done using the regression tool from the Data Analysis Toolkit in Excel or using the following simple formulas:

$$beta = \frac{corr_{i,m}\sigma_i}{\sigma_m} \tag{2}$$

$$alpha = \bar{r}_i - beta\ \bar{r}_m \tag{3}$$

The regression analysis will provide additional statistical information on the relationship, but the above equations are sufficient.

[5] Market beta regressions are often run using returns in excess of a risk-free rate and are often also adjusted for various statistical problems; however, this simple form typically provides very similar results.

- Based on the betas, which firm is the most risky? Least risky? How does your answer compare with the answer you provided based on standard deviations? Which one is appropriate for our hypothetical investors? Explain why.
- What is the economic meaning of the alpha?

Step 6: Since the beta provides a measure of risk, one should be able to link expected return to this risk measure. By expected return, we mean the return that an individual investor would expect to earn from holding this stock. This can be calculated quite easily. First, one recognizes that a riskless bond has a beta of zero and that the market portfolio has a beta of one. Since all the diversification benefits associated with portfolios have already been accounted for, the graph of the relationship between beta and returns for portfolios combining the risk-free bond and the market portfolio is linear. All individual stocks must lie on this same line. Thus, the equation for returns for every security is the equation of the line that connects the market return and the bond return in a graph of returns (vertical axis) against beta (horizontal axis).[6] That equation and the theories that justify it are referred to as the CAPM). The equation is

$$r_i = r_{rf} + \beta\,(r_m - r_{rf}) = r_{rf} + \beta\,(MRP) \tag{4}$$

where r_i is the expected return for an individual stock, r_{rf} is the expected risk-free rate of return, β is the beta, r_m is the expected return on the market, and (in the simplified version) *MRP* is the market risk premium (the difference between the expected return on the market and the expected risk-free rate of return).

Given the calculated beta for each stock, calculate the expected monthly return for each stock assuming that the *monthly* risk-free rate is 0.45% (about 5.5% annually) and the *monthly* risk premium is 0.50% (about 6% annually). Calculate the *annual* expected returns implied by these monthly returns. Calculate the realized annual returns from the mean returns in Step 1.

- How do the realized returns compare with the expected returns? Assuming the CAPM describes the appropriate expected returns for these stocks, describe how prices might respond if, at some point, the expected returns on the three stocks differed from what was predicted by the CAPM.

Summary

These steps illustrate the essential link between risk and returns as it would be viewed in a world where individuals can (and therefore do) hold portfolios of investments. We have used past data to generate a description of the world and considered our choices assuming the past behavior would indicate future (expected) behavior. There are a number of reasons one needs to be cautious

[6] It is crucial to recognize that this graph is plotting returns relative to beta while the earlier graphs plot returns relative to standard deviations.

about this assumption, and in many cases the results for past data would not be a reasonable estimate for the future. Applying the ideas developed here should be based on best estimates of future expected return behavior.